A New Coupling Construction for Markov Chains in Random Environments

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Let E and F be complete separable metric spaces equipped with their Borel σ-fields $\mathcal{B}(E)$ and $\mathcal{B}(F)$, respectively. Let $(X_n)_{n\in\mathbb{Z}}$ be a strictly stationary F-valued process. We consider the recursively defined process

$$Y_{n+1} = f(Y_n, X_n, \varepsilon_{n+1}), \qquad n \in \mathbb{N},$$

where Y_0 is a possibly random initial state, $f : E \times F \times \mathbb{R}^d \to E$ is measurable, and $(\varepsilon_n)_{n\geq 1}$ is a $[0,1]$-valued noise sequence. We assume that $(\varepsilon_n)_{n\geq 1}$ is i.i.d., independent of $(X_n)_{n\in\mathbb{Z}}$, and that Y_0 is independent of $(\varepsilon_n)_{n\geq 1}$. Then $(Y_n)_{n\in\mathbb{N}}$ forms a Markov chain in a random environment (MCRE): conditionally on the exogenous process $(X_n)_{n\in\mathbb{Z}}$, the process $(Y_n)_{n\in\mathbb{N}}$ is a time-inhomogeneous Markov chain whose transition probabilities are modulated by the random environment.

Markov chains in random environments have been extensively studied under Foster–Lyapunov-type drift and minorization conditions, adapted from the theory of general state-space Markov chains (see, e.g., [1], [3] and references therein). These assumptions ensure the existence of suitable small sets in the state space that are visited sufficiently often by the process, and on which coupling occurs with positive probability at each visit.

In this talk, we present an alternative coupling approach, developed in detail in [2]. Instead of relying on the existence of small sets, we assume that the dynamics is contractive at large distances. Moreover, we replace the classical minorization condition by a weaker local version, which guarantees that trajectories of $(Y_n)_{n\in\mathbb{N}}$ that come sufficiently close to each other couple with positive probability.

Under a few additional mild technical assumptions, we establish the existence of a stationary solution $\big((X_n, Y_n^*)\big)_{n\in\mathbb{Z}}$. Furthermore, we prove that the law of (X_n, Y_n) converges in total variation distance to the stationary law $\mathrm{Law}((X_0, Y_0^*))$, and we provide explicit convergence rate estimates. Finally, we derive explicit upper bounds on the α-mixing coefficients of the joint process $(X_n, Y_n)_{n\in\mathbb{N}}$ in terms of the α-mixing coefficients of the environment.

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