Deterministic and Monte Carlo Approximation scheme for Stochastic Differential Equations with Interaction

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We consider the stochastic differential equation with interaction introduced by A. A. Dorogovtsev [1, 2]:

$$\begin{aligned} dx(u,t) &= a\big(x(u,t),\mu_t\big)\,dt + \int_{\mathbb{R}^d} b\big(x(u,t),\mu_t,p\big)\,W(dp,dt),\\ x(u,0) &= u, \qquad u\in\mathbb{R}^d, \qquad \mu_t = \mu_0\circ x(\cdot,t)^{-1}. \end{aligned} \tag{1}$$

Here W is a Wiener sheet on $\mathbb{R}^d\times[0,+\infty)$,

$$a:\mathbb{R}^d\times M_2(\mathbb{R}^d)\to\mathbb{R}^d, \qquad b:\mathbb{R}^d\times M_2(\mathbb{R}^d)\times\mathbb{R}^d\to\mathbb{R}^{d\times d},$$

and $p\mapsto b(x,\mu,p)$ belongs to $L_2(\mathbb{R}^d;\mathbb{R}^{d\times d})$ for every (x,μ). For each u, the process $x(u,\cdot)$ is the trajectory of the particle starting from u, while μ_t is the image of the initial mass distribution under the random map $u\mapsto x(u,t)$. Thus, the coefficients at time t depend on both the position of an individual particle and the current mass distribution of the entire system.
For $p\geq 1$, let $M_p(\mathbb{R}^d)$ denote the space of all Borel probability measures μ on $\mathbb{R}^d$ with finite pth moment,

$$\int_{\mathbb{R}^d}|z|^p\,\mu(dz)<\infty.$$

For $\mu,\nu\in M_p(\mathbb{R}^d)$, the Wasserstein distance of order p is defined by

$$\gamma_p(\mu,\nu):=\left(\inf_{\varkappa\in C(\mu,\nu)}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^p\,\varkappa(dx,dy)\right)^{1/p}, \tag{2}$$

where $C(\mu,\nu)$ is the set of all couplings of μ and ν [1, 2].
We impose the following global Lipschitz assumptions: there exists $L>0$ such that, for all $x,y\in\mathbb{R}^d$ and $\mu,\nu\in M_2(\mathbb{R}^d)$,

$$|a(x,\mu)-a(y,\nu)|\leq L\big(|x-y|+\gamma_2(\mu,\nu)\big), \tag{3}$$

$$\int_{\mathbb{R}^d}|b(x,\mu,p)-b(y,\nu,p)|^2\,dp\leq L^2\big(|x-y|^2+\gamma_2^2(\mu,\nu)\big). \tag{4}$$

Under these assumptions, the stochastic differential equation with interaction admits a unique solution on $[0,+\infty)$. Moreover, for every $t>0$, the measure $\mu_t=\mu_0\circ x(\cdot,t)^{-1}$ is a random element in $\mathfrak{M}_2(\mathbb{R}^d)$ [2].
Let's consider the following examples of equations with interaction:
Kuramoto system [8]
Let

$$\mathbb{T}:=\mathbb{R}/(2\pi\mathbb{Z})$$

be the one-dimensional torus. Since each oscillator is characterized by its phase $\vartheta_i(t)\in\mathbb{T}$ and its fixed frequency $\omega_i\in\mathbb{R}$, we consider the state space

$$E:=\mathbb{T}\times\mathbb{R}, \qquad z_i(t):=(\vartheta_i(t),\omega_i).$$

Define the empirical measure

$$\mu_t^N := \frac{1}{N}\sum_{j=1}^{N}\delta_{z_j(t)} = \frac{1}{N}\sum_{j=1}^{N}\delta_{(\vartheta_j(t),\omega_j)}. \tag{5}$$

The Kuramoto system in integral form is

$$\vartheta_i(t) = \vartheta_i(0) + \int_0^t \left[\omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin\big(\vartheta_j(s) - \vartheta_i(s)\big)\right] ds. \tag{6}$$

Equivalently, it can be written as an equation with interaction

$$z_i(t) = z_i(0) + \int_0^t A\big(z_i(s), \mu_s^N\big)\, ds, \tag{7}$$

where

$$A((\vartheta,\omega),\mu) := \left(\omega + K\int_E \sin(\vartheta' - \vartheta)\,\mu(d\vartheta', d\omega'),\, 0\right). \tag{8}$$

Helmholtz point-vortex system [9]
For vortex positions $r_i(t) \in \mathbb{R}^2$ and intensities k_i, The integral form of the particle system is

$$r_i(t) = r_i(0) - \frac{1}{\pi}\int_0^t \sum_{j\neq i} k_j \nabla \log|r_i(s) - r_j(s)|\, ds. \tag{9}$$

Equivalently,

$$r_i(t) = r_i(0) + \int_0^t A(r_i(s), \mu_s)\, ds, \tag{10}$$

with

$$A(v,\mu) := -\frac{1}{\pi}\int_{\mathbb{R}^2} \mathbf{1}_{\{y\neq v\}} \nabla \log|v - y|\,\mu(dy). \tag{11}$$

Equation (10) has the measure-dependent structure of an equation with interaction.
Vortex filament and Landau–Lifshitz equations [10]
Let $\Gamma = \Gamma(s,t) \in \mathbb{R}^3$ be a filament parametrized by arc length, $|\partial_s\Gamma(s,t)| = 1$. The vortex-filament equation is

$$\Gamma(s,t) = \Gamma(s,0) + \int_0^t \partial_s\Gamma(s,\tau)\times\partial_{ss}\Gamma(s,\tau)\, d\tau. \tag{12}$$

Equivalently, since $\partial_s\Gamma \times \partial_{ss}\Gamma = \kappa B$, one has $\partial_t\Gamma = \kappa B$, where κ and B are the curvature and the binormal vector.
If

$$T(s,t) := \partial_s\Gamma(s,t), \qquad |T(s,t)| = 1, \tag{13}$$

then differentiation of (12) with respect to s gives the isotropic Landau–Lifshitz equation

$$T(s,t) = T(s,0) + \int_0^t T(s,\tau)\times\partial_{ss}T(s,\tau)\, d\tau. \tag{14}$$

[6, 7].
A natural approximation strategy for equation (1) is to discretize both the spatial dependence of the initial measure and the time variable. For compactly supported initial distributions, this can be achieved by replacing μ_0 with a finitely supported measure on a regular spatial grid and then applying the Euler–Maruyama scheme in time. However, the number of grid nodes grows rapidly with the dimension of the state space. This motivates replacing the deterministic spatial discretization by an empirical Monte Carlo measure, which requires only N

sampled support points and remains applicable to initial distributions with unbounded support under suitable moment assumptions.

1. Deterministic spatial and temporal approximation. Assume

$$\operatorname{supp}\mu_0 \subset K = [-R,R]^d. \tag{15}$$

For $N \in \mathbb{N}$, let

$$h_N := \frac{2R}{N}$$

and introduce the regular Cartesian grid

$$\mathcal{G}_N := \left\{ u_{\mathbf{i}}^N = (-R + i_1 h_N, \dots, -R + i_d h_N) : \mathbf{i} = (i_1,\dots,i_d) \in \{0,\dots,N\}^d \right\}. \tag{16}$$

Thus, the grid size is $h_N = 2R/N$ and the grid contains $(N+1)^d$ nodes.
For $u \in K$, let

$$\pi_N(u) := \operatorname*{arg\,min}_{v \in \mathcal{G}_N} |u - v|, \qquad \widetilde{\mu}_0^N := \mu_0 \circ \pi_N^{-1}. \tag{17}$$

Let $h = n^{-1}$, $t_k = kh$, and

$$\kappa_n(s) := t_k, \qquad s \in [t_k, t_{k+1}). \tag{18}$$

The deterministic-grid Euler–Maruyama approximation is

$$\begin{aligned} x_n^N(u,t) = u &+ \int_0^t a\big(x_n^N(u,\kappa_n(s)), \mu_{\kappa_n(s)}^{N,n}\big)\, ds \\ &+ \int_0^t \int_{\mathbb{R}^d} b\big(x_n^N(u,\kappa_n(s)), \mu_{\kappa_n(s)}^{N,n}, p\big)\, W(dp,ds), \end{aligned} \tag{19}$$

where

$$\mu_t^{N,n} := \widetilde{\mu}_0^N \circ \big(x_n^N(\cdot,t)\big)^{-1}. \tag{20}$$

Equivalently, at the grid times,

$$\begin{aligned} x_{n,k+1}^N(u) = x_{n,k}^N(u) &+ h\, a\big(x_{n,k}^N(u), \mu_k^{N,n}\big) \\ &+ \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^d} b\big(x_{n,k}^N(u), \mu_k^{N,n}, p\big)\, W(dp,ds), \end{aligned} \tag{21}$$

with

$$x_{n,0}^N(u) = u, \qquad \mu_k^{N,n} = \widetilde{\mu}_0^N \circ (x_{n,k}^N)^{-1}. \tag{22}$$

Theorem 2 *Let* (15) *and* (3)–(4) *hold. Fix* $p > d$,

$$0 < \eta < \frac{p-d}{p}, \qquad \sigma := \frac{\eta}{\eta + d}. \tag{23}$$

Then there exists $C < \infty$, *independent of* N *and* n, *such that*

$$\begin{aligned} &\mathbb{E} \sup_{u \in K} \sup_{t \in [0,1]} |x(u,t) - x_n^N(\pi_N(u),t)| \\ &\qquad \le C \left((\gamma_2^2(\mu_0, \widetilde{\mu}_0^N))^\sigma + N^{-\eta} + n^{-1/2} \right). \end{aligned} \tag{24}$$

then

$$\mathbb{E} \sup_{u \in K} \sup_{t \in [0,1]} |x(u,t) - x_n^N(\pi_N(u),t)| \le C \left(N^{-2\sigma} + N^{-\eta} + n^{-1/2} \right). \tag{25}$$

2. Monte Carlo approximation. To reduce the number of spatial nodes required by the discretization procedure and to avoid the dimensional dependence of regular grid constructions, we propose to approximate the initial measure by an empirical Monte Carlo measure

The deterministic measure (17) is replaced by

$$\mu_0^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{U_i}, \qquad U_1,\dots,U_N \text{ i.i.d. with law } \mu_0, \qquad (U_1,\dots,U_N)\perp W. \tag{26}$$

The Euler–Maruyama–Monte Carlo approximation is

$$\begin{aligned} x_n^N(u,t) = u &+ \int_0^t a\big(x_n^N(u,\kappa_n(s)),\mu_{\kappa_n(s)}^{N,n}\big)\,ds \\ &+ \int_0^t\int_{\mathbb{R}^d} b\big(x_n^N(u,\kappa_n(s)),\mu_{\kappa_n(s)}^{N,n},p\big)\,W(dp,ds), \end{aligned} \tag{27}$$

where

$$\mu_t^{N,n} := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_n^N(U_i,t)}. \tag{28}$$

At $t_k = kh$,

$$\begin{aligned} x_{n,k+1}^N(u) = x_{n,k}^N(u) &+ h\,a\big(x_{n,k}^N(u),\mu_k^{N,n}\big) \\ &+ \int_{t_k}^{t_{k+1}}\int_{\mathbb{R}^d} b\big(x_{n,k}^N(u),\mu_k^{N,n},p\big)\,W(dp,ds), \end{aligned} \tag{29}$$

with

$$x_{n,0}^N(u) = u, \qquad \mu_k^{N,n} = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_{n,k}^N(U_i)}. \tag{30}$$

Assume

$$\mu_0 \in M_q(\mathbb{R}^d), \qquad q>2, \tag{31}$$

and

$$q\neq 4 \quad (d\le 4), \qquad q\neq\frac{2d}{d-2} \quad (d>4). \tag{32}$$

Define

$$R_{d,q}(N) := \begin{cases} N^{-1/2}+N^{-(q-2)/q}, & d<4,\\ N^{-1/2}\log(1+N)+N^{-(q-2)/q}, & d=4,\\ N^{-2/d}+N^{-(q-2)/q}, & d>4. \end{cases} \tag{33}$$

Theorem 3 *Under* (31)–(32),

$$\mathbb{E}\,\gamma_2^2(\mu_0,\mu_0^N) \le C(d,q,M_q(\mu_0))R_{d,q}(N). \tag{34}$$

For $\theta\in(0,1)$, set

$$\alpha := \frac{\theta}{2\theta+d}. \tag{35}$$

The two components of the total error satisfy

$$\mathbb{E}\sup_{u\in K}\sup_{t\in[0,1]}|x(u,t)-x^N(u,t)| \le C_K R_{d,q}(N)^{\alpha}, \tag{36}$$

$$\mathbb{E}\sup_{u\in K}\sup_{t\in[0,1]}|x^N(u,t)-x_n^N(u,t)| \le C_K n^{-\alpha}. \tag{37}$$

Theorem 4 *Let* (3)–(4) *and* (31)–(32) *hold. Let $K\subset\mathbb{R}^d$ be compact and let α be given by* (35). *Then*

$$\begin{aligned} &\mathbb{E}\sup_{u\in K}\sup_{t\in[0,1]}|x(u,t)-x_n^N(u,t)| \\ &\quad\le \mathbb{E}\sup_{u,t}|x(u,t)-x^N(u,t)| + \mathbb{E}\sup_{u,t}|x^N(u,t)-x_n^N(u,t)| \\ &\quad\le C_K\left(R_{d,q}(N)^{\alpha}+n^{-\alpha}\right). \end{aligned} \tag{38}$$

The passage from the first result to the second is expressed by

$$\widetilde{\mu}_0^N = \mu_0 \circ \pi_N^{-1} \quad \longrightarrow \quad \mu_0^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{U_i}, \tag{39}$$

$$\operatorname{supp} \mu_0 \subset [-R, R]^d \quad \longrightarrow \quad \mu_0 \in M_q(\mathbb{R}^d), \qquad q > 2, \tag{40}$$

while the total error changes from

$$C\left((\gamma_2^2(\mu_0, \widetilde{\mu}_0^N))^\sigma + N^{-\eta} + n^{-1/2}\right) \tag{41}$$

to

$$C_K\left(R_{d,q}(N)^\alpha + n^{-\alpha}\right). \tag{42}$$

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